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BRIEF COMMUNICATION

A NOTE ON TWO-PHASE SEPARATED FLOW MODELS

J. C. ROUSSEAU
Commissariat à L'Energie Atomique, Centre d'Etudes Nucléaires de Grenoble, 38041 Grenoble, France

and

R. L. FERCH
Atomic Energy of Canada Ltd., Whiteshell Nuclear Research Establishment, Pinawa, Manitoba, Canada R0E 1L0

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There is considerable interest in the development of one-dimensional mathematical models for two-phase flow in which the velocities of the two phases are unequal. Many such models have included among their assumptions that of equality of the two phase pressures. In general, the system of equations obtained in these models exhibit non-hyperbolic behavior, signifying that the system is unstable to short-wavelength pertubations. The mathematical initial value problem for such a non-hyperbolic system is ill-posed, and it is not clear what the correct boundary condition specifications are (if any such exist) in such a case. Therefore, models which are hyperbolic in structure for the flow regimes of interest are sought, in the interests of simplicity, and of a better understanding of how to solve problems and interpret the results.

One such approach which has been presented recently (Banerjee *et al.* 1978; Mathers *et al.* 1978; Agee *et al.* 1978) allows the pressures in the two phases to vary independently. The dependent variables in this "UVUTUP" model are seven in number: u_1, h_1, p_1, u_2, h_2, p_2 and α. The required differential equations are obtained from the conservation equations for mass, energy and axial momentum in each phase, together with the interfacial jump conditions for these three quantities. Of the three interfacial equations, one is algebraic, and the other two contain only one differential term each (assuming the various interfacial terms m_{kI}, τ_{kI}, q_{kI}, u_{kI}, h_{kI}, p_kI to be algebraic in form), namely $(p_{1I} - p_{2I})\partial\alpha/\partial x$ and $(p_{1I} - p_{2I})\partial\alpha/\alpha t$ respectively. Here p_{kI} is the pressure in phase k at the interface. One combines these last two equations to obtain the seventh, void propagation equation. This is, in effect, achieved by eliminating one (or one combination) of the interfacial transfer terms from the system. The resultant system is essentially one of two quasi-independent, single-phase flows in variable-area channels, connected through the seventh, void propagation equation.

This method has been criticized on a number of points. Firstly, the combination used to obtain the void propagation equation is not unique; any arbitrary combination of the two jump conditions apparently will serve. Secondly, the void propagation velocity obtained in this way does not appear to be physically meaningful. Thirdly, objections have been raised to the use of the interfacial jump equations to close the system; logically, these should be considered as constraints on the interfacial transfer rates, and the information required to close the system should come from independent physical arguments (Bouré 1978).

We outline here one possible method for closing the equation set which is not subject to these criticisms. The resultant set of equations is six in number, and will be seen to be hyperbolic provided the assumptions of the physical closure model are satisfied.

We begin by writing the cross-sectionally averaged conservation equations for each phase, and jump conditions, as:

$$\frac{\partial}{\partial t}\alpha_k\rho_k+\frac{\partial}{\partial x}\alpha_k\rho_k u_k=\Gamma^m_{kI} \qquad [1]$$

$$\frac{\partial}{\partial t}\alpha_k\rho_k u_k+\frac{\partial}{\partial x}\alpha_k\rho_k u_k^2+\alpha_k\frac{\partial p_k}{\partial x}+(p_k-p_{kI})\frac{\partial \alpha_k}{\partial x}=\Gamma^f_{kI}+\Gamma^f_{kw} \qquad [2]$$

$$\frac{\partial}{\partial t}\alpha_k\rho_k E_k+\frac{\partial}{\partial x}\alpha_k\rho_k u_k E_k-\alpha_k\frac{\partial p_k}{\partial t}-(p_k-p_{kI})\frac{\partial \alpha_k}{\partial t}=\Gamma^e_{kI}+\Gamma^e_{kw} \qquad [3]$$

$$\sum_{k=1}^{2}\Gamma^m_{kI}=0 \qquad [4]$$

$$\sum_{k=1}^{2}\Gamma^f_{kI}+p_{kI}\frac{\partial\alpha_k}{\partial x}=0 \qquad [5]$$

$$\sum_{k=1}^{2}\Gamma^e_{kI}-p_{kI}\frac{\partial\alpha_k}{\partial t}=0 \qquad [6]$$

where $E_k=h_k+1/2\,u_k^2$, subscript I refers to the interface, and subscript w to wall and external source terms. The Γ_{kw} and Γ_{kI} terms contain the wall and interface mass, momentum and energy transfer rates. We have assumed, for simplicity, constant cross-sectional area, and have neglected derivatives of τ_k and q_k. Surface tension has been neglected in [4]–[6]. We shall also assume that the Γ_{kI} terms are algebraic in form (i.e. they do not depend on derivatives of the dependent variables). The transverse momentum equations, in the same notation, are

$$\Gamma^t_{kI}+\Gamma^t_{kw}=0. \qquad \text{and} \qquad \sum_{k=1}^{2}\Gamma^t_{kI}=0\,.$$

If we search for physically derived equations to close the system, one possible choice is the equations of conservation of transverse momentum, especially in cases (e.g. stratified flow) where there is asymmetry of the flow in the transverse direction. Considering first the interfacial transverse momentum jump condition, we find that by neglecting the terms corresponding to the interfacial stress difference (in which case $\Gamma^t_{kI}=(-1)p_{kI}$), this equation reduces to $p_{1I}=p_{2I}(=p_I)$. That is, the pressures in the two phases are equal at the interface. This immediately reduces the other jump equations [5] and [6] to purely algebraic form, so that all four jump equations may be considered simply as constraints on the interfacial transfer terms.

The remaining closure information comes from the transverse momentum conservation equations in each phase. Neglecting inertial (derivative) terms, these are written in the form of simple force balances. These relate the external forces, interfacial and wall stress terms and the pressures at the wall and the interface, through coefficients dependent on the geometry. By making suitable assumptions about the geometrical coefficients and the pressure distribution across each phase, we may write these equations to define p_k-p_I in terms of the dependent flow variables. These equations will, of course, be specific to the flow pattern assumed. Furthermore, by subtracting these equations, we obtain an expression of the form

$$p_1-p_2=f(u_1,u_2,h_1,h_2,p_1,p_2,\alpha)\,. \qquad [7]$$

This expression, together with [1]–[3], the state equations, and appropriate models for the Γ terms, forms a closed system of equations with six degrees of freedom.

This procedure might be applied to a variety of two-phase flow problems. The simplest such case is that of horizontal stratified flow in a rectangular channel. In this case, $p_1 - p_I = -\alpha\rho_1 gH/2$ and $p_2 - p_I = (1-\alpha)\rho_2 gH/2$, where H is the height of the channel, and phase 1 is the vapour (low-density) phase. That is, the pressure drop across each phase is equal to the weight of the phase in cross section. This hydrostatic assumption will, of course, fail for high-velocity flows, as does the assumption of a nearly flat stratified flow. The resulting equations are similar to those of the UVUTEP model used by a number of investigators, with the addition of the $(p_k - p_I)\partial\alpha_k/\partial x$ and $(p_k - p_I)\partial\alpha_k/\partial t$ terms. These terms must be included if the tendency of liquid to flow towards the lowest available point is to be described. The pressures in the two phases differ by the gravitational "head" $[\alpha\rho_1 + (1-\alpha)\rho_2]gH/2$. This introduces a number of new terms into the equations, due to the dependence of f on h_1, h_2, p_1, p_2 and α. However, if we assume that $\sqrt{(gH)}$ is small compared to the sound speed in either phase, most of these terms may be neglected; only the dependence of f on α is significant.

The approximate equations derived in this way are of the same form as [1]–[3], where we make the substitutions $p_2 = p_1$, $p_1 - p_{1I} = -\alpha\rho_1 gH/2$, and $p_2 - p_{2I} = (1-\alpha)(2\rho_2 - \rho_1)gH/2$. A number of terms, of order gH/a_k^2 (where a_k^2 is the sound speed in phase k), must be neglected in order to obtain this simplified form.

We have performed a characteristic analysis for the more "exact" model, in which the neglected terms mentioned above have been retained. The characteristics are the roots of the polynomial equation

$$(\lambda - u_1)(\lambda - u_2)[A(\lambda - u_1)^2(\lambda - u_2)^2 + B(\lambda - u_2)^2 + C(\lambda - u_2)^2 + D] = 0 \qquad [8]$$

where

$$A = (1-\alpha)\rho_1 a_1^2 + \alpha\rho_2 a_2^2 + \alpha(1-\alpha)\left[a_2^2\frac{\partial\rho_2}{\partial h_2} - a_1^2\frac{\partial\rho_1}{\partial h_1}\right]\frac{gH}{2}$$

$$B = -(1-\alpha)\left[\rho_1 a_1^2 a_2^2 + \alpha a_2^2\left(2\rho_2 - a_1^2\frac{\partial\rho_1}{\partial h_1}\right)\frac{gH}{2} - \alpha(1-\alpha)\left(\rho_2 - a_2^2\frac{\partial\rho_2}{\partial h_2}\right)\left(\frac{gH}{2}\right)^2\right]$$

$$C = -\alpha\left[\rho_2 a_1^2 a_2^2 + (1-\alpha)a_1^2\left(2\rho_1 - a_2^2\frac{\partial\rho_2}{\partial h_2}\right)\frac{gH}{2} - \alpha(1-\alpha)\left(\rho_1 - a_1^2\frac{\partial\rho_1}{\partial h_1}\right)\left(\frac{gH}{2}\right)^2\right]$$

$$D = \alpha(1-\alpha)\left[a_1^2 a_2^2(\rho_2 - \rho_1)gH - \left\{\alpha a_2^2\left(\rho_1 - a_1^2\frac{\partial\rho_1}{\partial h_1}\right) - (1-\alpha)a_1^2\left(\rho_2 - a_2^2\frac{\partial\rho_2}{\partial h_2}\right)\right\}\left(\frac{gH}{2}\right)^2\right].$$

We may make use of the fact that $gH \ll a_k^2$ to obtain the following approximations to the roots:

$$\lambda = u_1, u_2, u' \pm V, u^* \pm a^*$$

where

$$u' = [(1-\alpha)\rho_1 u_1 + \alpha\rho_2 u_2]/\rho^*$$

$$u^* = [(1-\alpha)\rho_1 u_2 + \alpha\rho_2 u_1]/\rho^*$$

$$V^2 = \frac{(1-\alpha)}{\rho^*}\left[(\rho_2 - \rho_1)gH - \frac{\rho_1\rho_2}{\rho^*}(u_1 - u_2)^2\right]$$

$$\frac{1}{a^{*2}} = \frac{1}{\rho^*}\left[\frac{(1-\alpha)\rho_1}{a_2^2} + \frac{\alpha\rho_2}{a_1^2}\right]$$

and

$$\rho^* = (1-\alpha)\rho_1 + \alpha\rho_2$$

(The approximate characteristic roots for the simplified equations described above are identical with these, to this order of approximation.)

The last pair of roots, $u^* \pm a^*$, represent the propagation of sound waves. The mixture sound speed a^* is not the same as those obtained in the cases of the equilibrium and the equal velocity unequal-temperature models; $a_1 < a^* < a_2$, and in fact, except at very low α, $a^* \sim a_1$. This appears to be in good agreement with experimental measurements for stratified and annular flows and is consistent with that presented elsewhere (Wallis 1969), but is in sharp contrast with the much lower sound speed obtained in the homogeneous equilibrium model, which is applicable to strongly mixed flows. The translation velocity associated with these sound waves, u^*, is also very nearly equal to u_1. Thus the propagation of sound waves is dominated by the vapour.

The other pair of roots, $u' \pm V$, represent the propagation of surface waves. Here $u' \sim u_2$, and as expected, the liquid phase dominates. We note that V is real, and therefore that the system is hyperbolic, only when

$$(u_1 - u_2)^2 < \left(\frac{\alpha}{\rho_1} + \frac{1-\alpha}{\rho_2}\right)(\rho_2 - \rho_1)gH. \qquad [10]$$

This is the long wavelength limit of the classical Kelvin–Helmholtz criterion (Milne–Thomson 1960, p. 405) for the stability of the interface. Note that the difference between the long- and short-wavelength limits is due to the fact that the assumption of linear variation in pressure at a cross section is valid only when the height is small compared to the wavelength. Therefore the assumptions made in deriving [8] are only adequate for long waves. It is however, interesting to note that the model fails to be hyperbolic when the physical assumptions of the model (i.e. almost flat stratified flow) fail. The inadequacy of the model in describing short waves may perhaps be less serious than it appears, since for thermalhydraulic computations one is interested mainly in the longer scale lengths, and a volume-averaging process is either explicitly or implicitly performed to remove short scale length variability.

At the point at which this stratified flow model becomes non-hyperbolic, the dispersion relation for small-amplitude sinusoidal perturbations predicts that short wavelength surface waves will grow exponentially in amplitude. (Note that for wavelengths short compared to the typical gradient scale length, the dispersion relation is identical with the characteristic equation [8].) We might perhaps interpret this instability as representing a transition between two flow patterns. As an example, if when the constitutive terms, and/or steep gradients in the flow variables, are taken into account, the waves which grow most rapidly are of long wavelength. then we might expect the flow pattern to change to intermittent or slug flow. Such a more detailed dispersion analysis is of considerable interest, but is unfortunately rather ardous for hand computation.

For computational purposes, the simplified model, in which the only remaining effect of the unequal phase pressures appears in the $(p_k - p_{kI})$ terms, provides virtually identical results to the more complete version. This points up the fact that the significant new feature of the UVUTUP model is not that the phase pressures are unequal, but that their derivatives are unequal, as reflected in this case by these $(p_k - p_{kI})$ terms. The practical utility of this particular model is probably small, since flat stratified flows are of limited interest. However, it is possible that it might be made more useful through the addition of other derivative terms (e.g. "virtual mass" terms), to allow the description of flows with large amplitude waves present.

In conclusion, one method of closing the equation set, which describes stratified two-phase flows, using information from the equations of conservation of transverse momentum, has been presented. This approach answers some of the objections raised to the UVUTUP system, while retaining most of its major features. It is hoped that similar methods can be applied to other flow regimes, such as, for example, dispersed flows (Stuhmiller 1977). A more detailed investigation, involving a wider variety of flow regimes, is planned.

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